A LOWER BOUND FOR THE SIZE OF A SUM OF DILATES

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Abstract. Let A be a subset of integers and let $2 \cdot A + k \cdot A = \{2a_1 + ka_2 : a_1, a_2 \in A\}$. Y. O. Hamidoune and J. Rué proved in [5] that if k is an odd prime and A a finite set of integers such that $|A| > 8k^k$, then $|2 \cdot A + k \cdot A| \geq (k+2)|A| - k^2 - k + 2$. In this paper, we extend this result for the case when k is a power of an odd prime and the case when k is a product of two odd primes.

1. Introduction

Let k be an integer and let A be a finite set of integers. The k-dilation $k \cdot A$ of the set A is the set of all integers of the form ka, where $a \in A$. Let $f(x_1, \ldots, x_n) = u_1x_1 + \cdots + u_nx_n$ be a linear form with integer coefficients $u_1, \ldots, u_n$. We define the set $f(A) = u_1 \cdot A + \cdots + u_n \cdot A = \{u_1a_1 + \cdots + u_ha_n : a_i \in A\}$. B. Bukh, in [1] obtained the almost sharp lower bound for the size of the sets $f(A)$: $|u_1 \cdot A + \cdots u_n \cdot A| \geq (|u_1| + \cdots + |u_n|)|A| - o(|A|)$, where $u_1, \ldots, u_n$ are integers such that $(u_1, \ldots, u_n) = 1$.

In the case of binary linear forms we write $f(x, y) = mx + ky$, where m and k are nonzero integers. We are interested in finding a sharp lower bound for $|f(A)|$. It is easy to see ([7]) that it is enough to consider only normalized binary linear forms satisfying $k \geq |m| \geq 1$ and $(m, k) = 1$. Many authors ([1],[2],[3],[8]) studied the lower bounds of $|f(A)|$ for the case $m = 1$. The sharp lower bound for $|A + k \cdot A|$ was known for the case $k = 1$ (see [6]), and it was given for $k = 2$ in [8] and $k = 3$ in [3]. J. Cilleruelo, M. Silva, C. Vinuesa conjectured in [3] that if k is a positive integer and A a finite set of integers with sufficiently large cardinality, then $|A + k \cdot A| \geq (k+1)|A| - \lceil k(k+2)/4 \rceil$. This conjecture was proved for the case when k is a prime number in [2], and very recently for the case when k is a power of a prime and k is a product of two primes in [4].

The case $m = 2$ was studied in [5]. Y. O. Hamidoune and J. Rué proved in [5] that if k is an odd prime and A a finite set of integers such that $|A| > 8k^k$, then $|2 \cdot A + k \cdot A| \geq (k+2)|A| - k^2 - k + 2$. In this paper, we extend this result for the case when k is a prime power and a product of two primes. More precisely, we prove the following theorems.

Theorem 1. *Let A be a finite set of integers such that $|A| > 8k^k$. If $k = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{Z}_{\geq 1}$, then*

$$|2 \cdot A + k \cdot A| \geq (k+2)|A| - k^2 - k + 2.$$

Theorem 2. *Let A be a finite set of integers such that $|A| > 8k^k$. If $k = pq$, where p and q are distinct odd primes, then*

$$|2 \cdot A + k \cdot A| \geq (k+2)|A| - k^2 - k + 2.$$

2. Notation and Preliminaries

Let A be a finite set of integers and let k be a positive integer. We define $\hat{A}$ to be the natural projection of the set A on $\mathbb{Z}/k\mathbb{Z}$ and $c_k(A) = |\hat{A}|$. Then, if $c_k(A) = j$, we denote by $A_1, A_2, \dots, A_j$ the distinct congruences classes of A modulo k. We assume that $|A_1| \geq |A_2| \geq \dots \geq |A_j|$. For every $1 \leq i \leq j$, we write $A_i = kX_i + u_i$, where $0 \leq u_i < k$. Let $E = \{1 \leq i \leq j \mid |X_i| < k\}$ and let $F = \{1 \leq i \leq j \mid |X_i| = k\}$. We define the sets $\Delta_{ii} = (2A_i + k \cdot A) \setminus (2A_i + k \cdot A_i)$ for $1 \leq i \leq j$.

Lemma 3 (Chowla, [6]). *Let $n \geq 2$ and let A and B be nonempty subsets of $\mathbb{Z}/n\mathbb{Z}$. If $0 \in B$ and $(b, n) = 1$ for all $b \in B \setminus \{0\}$, then*

$$|A + B| \geq \min\{n, |A| + |B| - 1\}.$$

The following proposition, as well as its corollaries and the following lemma are Proposition 3.2, Corollary 3.3, Corollary 3.4 and Lemma 4.1 from [5].

Proposition 4. *Let A and B be finite set of integers and let n and m be coprime integer. Then*

$$|n \cdot A + m \cdot B| \geq c_n(B)|A| + c_m(A)|B| - c_m(A)c_n(B).$$

Corollary 5. *Let $2 \leq n < m$ be coprime integers. Let A be a finite set of integers. Then $|n \cdot A + m \cdot B| \geq 4|A| - 4$.*

Corollary 6. *Let k be an odd integer. Let A be a finite set of integers such that $c_k(A) = k$. Then $|2 \cdot A + k \cdot A| \geq (k + 2)|A| - 2k$.*

Lemma 7. *Let A be a finite set of integers and let k be a positive integer. Then*

$$\sum_{i=1}^{j} \Delta_{ii} \geq j(j - 1).$$

In the proof of Theorem 2, we will use the following lemmas. They appear as Lemma 6 and Lemma 8 in [4].

Lemma 8. *Let k be a positive integer and let A be a nonempty subset of $\mathbb{Z}/k\mathbb{Z}$. Let α be a nonzero element in $\mathbb{Z}/k\mathbb{Z}$. We have $A + \alpha = A$ if and only if*

$$A = \bigcup_{\beta \in I} ((k, \alpha) \cdot \{0, 1, \dots, \frac{k}{(k, \alpha)} - 1\} + \beta)$$

for some nonempty set $I \subset \mathbb{Z}/(k, \alpha)\mathbb{Z}$ and $\frac{k}{(k,\alpha)} \mid |A|$.

Lemma 9. *Let $k > 2$ be an integer that is not a prime and let A be a nonempty subset of $\mathbb{Z}/k\mathbb{Z}$. Let $(q, k) \neq 1$ and $0 \in B \subset (\{0, \bar{q}\} \cup \{\bar{b} \mid (b, k) = 1\})$. If $|A + \{0, \bar{q}\}| \geq |A| + 1$, then*

$$|A + B| \geq \min(k, |A| + |B| - 1).$$

3. The case $k = p^\alpha$

Lemma 10. *Let A be a finite set of integers such that $\gcd(A) = 1$ and $0 \in A$. Let $k = p^\alpha$, where p is an odd prime number and $\alpha \in \mathbb{Z}_{\geq 1}$. If $|\Delta_{ii}| < |A_i|$, then $c_2(A_i) = 2$.*

Proof. Let us assume that $c_2(A_i) = 1$. Thus, A_i contains only even or only odd integers.

Let A_i contains only even integers. There exists an odd $a \in A$, since $\gcd(A) = 1$. Then

$$|\Delta_{ii}| = |(2 \cdot A_i + k \cdot A) \setminus (2 \cdot A_i + k \cdot A_i)| \geq |(2 \cdot A_i + ka)| = |A_i|,$$

a contradiction.

Similarly, if A_i contains only odd integers

$$|\Delta_{ii}| = |(2 \cdot A_i + k \cdot A) \setminus (2 \cdot A_i + k \cdot A_i)| \geq |(2 \cdot A_i + k0)| = |A_i|,$$

a contradiction. □

Lemma 11. *Let A be a finite set of integers such that $\gcd(A) = 1$. Let $k = p^\alpha$, where p is an odd prime number and $\alpha \in \mathbb{Z}_{\geq 1}$. Let $m = \min\{1 \leq i \leq j \mid p \nmid u_i\}$ and $i \in E \setminus \{m\}$.*

(i) *If $p \mid u_i$, then $|\Delta_{ii}| \geq |A_m|$.*
(ii) *If $u_l = 0$ and $p \nmid u_i$, then $|\Delta_{ii}| \geq |A_l|$.*

Proof. (i) We have

$$|\Delta_{ii}| = |(2 \cdot A_i + k \cdot A) \setminus (2 \cdot A_i + k \cdot A_i)| \geq |(2 \cdot X_i + A_m) \setminus (2 \cdot X_i + A_i)|. \tag{1}$$

On the other hand $(u_m - u_i, k) = 1$, so using Lemma 3 and that $|X_i| < k$, we obtain

$$|2 \cdot \hat{X}_i + \{0, u_m - u_i\}| \geq |\hat{X}_i| + 1,$$

thus

$$|(2 \cdot \hat{X}_i + u_m) \setminus (2 \cdot \hat{X}_i + u_i)| \geq 1. \tag{2}$$

Combining (1) and (2), we conclude

$$|\Delta_{ii}| \geq |A_m||(2 \cdot \hat{X}_i + u_m) \setminus (2 \cdot \hat{X}_i + u_i)| \geq |A_m|.$$

(ii) Similarly as in (i),

$$|\Delta_{ii}| = |(2 \cdot A_i + k \cdot A) \setminus (2 \cdot A_i + k \cdot A_i)| \geq |(2 \cdot X_i + A_l) \setminus (2 \cdot X_i + A_i)|.$$

We have $(u_l - u_i, k) = 1$, so

$$|(2 \cdot \hat{X}_i + u_l) \setminus (2 \cdot \hat{X}_i + u_i)| \geq 1$$

and

$$|\Delta_{ii}| \geq |A_l||(2 \cdot \hat{X}_i + u_l) \setminus (2 \cdot \hat{X}_i + u_i)| \geq |A_l|.$$

□

Lemma 12. *Let A be a finite set of integers. If $k = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{Z}_{\geq 1}$, then*

$$|2 \cdot A + k \cdot A| \geq (k + 2)|A| - 4k^{k-1}.$$

Proof. Let T be the set of integers t such that for every finite set $A \subset \mathbb{Z}$

$$|2 \cdot A + k \cdot A| \geq (t+2)|A| - 4k^{t-1}.$$

We will use induction to prove $k \in T$. By Corollary 5, we obtain that $2 \in T$. Let us assume that $2 \leq t \leq k$ and $t-1 \in T$. Let A be a finite set of integers.

Case 1. $\sum_{i \in E} |\Delta_{ii}| \geq \sum_{i \in E} |A_i|$

By Corollary 6, for every $i \in F$, we have $|2 \cdot A_i + k \cdot A_i| \geq (k+2)|A_i| - 2k$. On the other hand, if $i \in E$, using induction hypothesis we get $|2 \cdot A_i + k \cdot A_i| \geq (t+1)|A_i| - 4k^{t-2}$. Hence,

$$\begin{aligned}
|2 \cdot A + k \cdot A| &= \sum_{i \in E} |2 \cdot A_i + k \cdot A| + \sum_{i \in F} |2 \cdot A_i + k \cdot A| \\
&\geq \sum_{i \in E} (|2 \cdot A_i + k \cdot A| + |\Delta_{ii}|) + \sum_{i \in F} |2 \cdot A_i + k \cdot A_i| \\
&\geq \sum_{i \in E} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E} |\Delta_{ii}| + \sum_{i \in F} [(k+2)|A_i| - 2k] \\
&\geq \sum_{i \in E} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E} |A_i| + \sum_{i \in F} [(k+2)|A_i| - 2k] \\
&\geq (t+2)|A| - (4|E|k^{t-2} + 2|F|k) \geq (t+2)|A| - 4k^{t-1}
\end{aligned}$$

Case 2. $\sum_{i \in E} |\Delta_{ii}| < \sum_{i \in E} |A_i|$.

Without loss of generality we may assume that $\gcd(A) = 1$ and $0 \in A_1$. We define $n = \min\{i \in E \mid |\Delta_{ii}| < |A_i|\}$. By Lemma 10, we have $c_2(A_n) = 2$. Let $m = \min\{1 \leq i \leq j \mid p \nmid u_i\}$. By Lemma 11, we have that $|\Delta_{ii}| \geq |A_m|$ for all $i \in E$. Note that $m \neq n$.

We have $m > n$. For if $m < n$, by Lemma 11, we have that $|\Delta_{ii}| \geq |A_n|$ for all $i \in E$ such that $i \geq n$ and this leads to contradiction:

$$\begin{aligned}
\sum_{i \in E} |\Delta_{ii}| &= \sum_{i \in E, i < n} |\Delta_{ii}| + \sum_{i \in E, i \geq n} |\Delta_{ii}| \\
&\geq \sum_{i \in E, i < n} |A_i| + \sum_{i \in E, i \geq n} |A_n| \geq \sum_{i \in E} |A_i|.
\end{aligned}$$

Next, by the definition of m, we have $p \mid u_n, \ldots, p \mid u_{m-1}$, so $(u_n - u_m, k) = \cdots = (u_{m-1} - u_m, k) = 1$. Using Lemma 3, we obtain

$$|2 \cdot \hat{X}_m + \{0, u_n - u_m, \ldots, u_s - u_m\}| \geq \min\{k, |\hat{X}_m| + s - n + 1\}, \text{ for all } n \leq s \leq m-1.$$

Let $s = n$. If $|\hat{X}_m| < k$, we have

$$|(2 \cdot \hat{X}_m + u_n) \setminus (2 \cdot \hat{X}_m + u_m)| \geq 1$$

and

$$|(2 \cdot X_m + A_n) \setminus (2 \cdot X_m + A_m)| \geq |A_n|.$$

Now, let $n < s < m-1$ such that

$$|(2 \cdot X_m + (A_n \cup A_{n+1} \cup \ldots \cup A_s)) \setminus (2 \cdot X_m + A_m)| \geq |A_n| + |A_{n+1}| + \cdots + |A_s|$$

and let us assume that $|\hat{X}_m| + s - n + 2 \leq k$. We have

$$|(2 \cdot \hat{X}_m + \{u_n, \ldots, u_s\}) \setminus (2 \cdot \hat{X}_m + u_m)| \geq s - n + 1$$

and

$$|(2 \cdot \hat{X}_m + \{u_n, \dots, u_s, u_{s+1}\}) \setminus (2 \cdot \hat{X}_m + u_m)| \geq s - n + 2,$$

so

$$|(2 \cdot X_m + (A_n \cup A_{n+1} \cup \ldots \cup A_{s+1}) \setminus (2 \cdot X_m + A_m)| \geq |A_n| + |A_{n+1}| + \cdots + |A_{s+1}|.$$

We distinguish two subcases.

Case 2a. $|\hat{X}_m| + m - n \leq k$.

We have

$$\begin{aligned}|\Delta_{mm}| &= |(2 \cdot A_m + k \cdot A) \setminus (2 \cdot A_m + k \cdot A_m)| \\ &= |(2 \cdot X_m + A) \setminus (2 \cdot X_m + A_m)| \\ &\geq |(2 \cdot X_m + (A_n \cup A_{n+1} \cup \ldots \cup A_{m-1})) \setminus (2 \cdot X_m + A_m)| \\ &\geq |A_n| + |A_{n+1}| + \cdots + |A_{m-1}|.\end{aligned}$$

By Lemma 11, we have $|\Delta_{ii}| \geq |A_m|$, for all $i \in E \setminus \{m\}$, so

$$\begin{aligned}|2 \cdot A + k \cdot A| = &\sum_{i \in E \setminus \{m\}} |2 \cdot A_i + k \cdot A| + \sum_{i \in F \setminus \{m\}} |2 \cdot A_i + k \cdot A| + |2 \cdot A_m + k \cdot A| \\ \geq &\sum_{i \in E \setminus \{m\}} (|2 \cdot A_i + k \cdot A_i| + |\Delta_{ii}|) + \sum_{i \in F \setminus \{m\}} |2 \cdot A_i + k \cdot A_i| \\ &+ |2 \cdot A_m + k \cdot A_m| + |\Delta_{mm}| \\ \geq &\sum_{i \in E \setminus \{m\}} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E \setminus \{m\}} |\Delta_{ii}| + \sum_{i \in F \setminus \{m\}} [(k+2)|A_i| - 2k] \\ &+ (t+1)|A_m| - 4k^{t-2} + |A_n| + |A_{n+1}| + \cdots + |A_{m-1}| \\ \geq &\sum_{i \in E \cup \{m\}} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E \cup \{m\}} |A_i| + \sum_{i \in F \setminus \{m\}} [(k+2)|A_i| - 2k] \\ \geq &(t+2)|A| - 4(|E| + |F|)k^{t-2} \geq (t+2)|A| - 4k^{t-1}.\end{aligned}$$

Case 2b. $|\hat{X}_m| + m - n > k$.

In this case

$$|2 \cdot \hat{X}_m + \{0, u_n - u_m, \dots, u_{m-1} - u_m\}| = k$$

and

$$|(2 \cdot X_m + (A_n \cup \ldots \cup A_m)) \setminus (2 \cdot X_m + A_n)| \geq (k - |\hat{X}_m|)|A_m|. \tag{3}$$

On the other hand, we have $c_2(X_n) = c_2(A_n) = 2$, so by Proposition 4

$$|2 \cdot X_m + A_n| = |2 \cdot X_m + k \cdot X_n| \geq 2|X_m| + |\hat{X}_m|(|X_n| - 2) = 2|A_m| + |\hat{X}_m|(|A_n| - 2). \tag{4}$$

We have $|A_n| \geq |A_m|$. Thus, by (3) and (4),

$$\begin{aligned}|2 \cdot A_m + k \cdot A| &= |2 \cdot X_m + A| \\ &\geq |2 \cdot X_m + A_n| + |2 \cdot X_m + (A_n \cup \ldots \cup A_m)) \setminus (2 \cdot X_m + A_n)| \\ &\geq 2|A_m| + |\hat{X}_m|(|A_n| - 2) + (k - |\hat{X}_m|)|A_m| \\ &\geq (k+2)|A_m| + |\hat{X}_m|(|A_n| - |A_m|) - 2k.\end{aligned}$$

By the definition of m, we have $m \leq p^{\alpha-1} + 1$, so

$$|\hat{X}_m| > k - m + n \geq k - m + 1 \geq p^{\alpha} - p^{\alpha-1} \geq p^{\alpha-1} \geq m - 1.$$

Thus

$$|2\cdot A_m + k\cdot A| \geq (k+2)|A_m| + m(|A_n| - |A_m|) - 2k$$

and

$$\begin{aligned}
|2\cdot A + k\cdot A| &= \sum_{i\in E\setminus\{m\}} |2\cdot A_i + k\cdot A| + \sum_{i\in F\setminus\{m\}} |2\cdot A_i + k\cdot A| + |2\cdot A_m + k\cdot A| \\
&\geq \sum_{i\in E\setminus\{m\}} (|2\cdot A_i + k\cdot A_i| + |\Delta_{ii}|) + \sum_{i\in F\setminus\{m\}} |2\cdot A_i + k\cdot A_i| \\
&\quad + (k+2)|A_m| + m(|A_n| - |A_m|) - 2k \\
&\geq \sum_{i\in E\setminus\{m\}} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i\in E\setminus\{m\}} |\Delta_{ii}| + \sum_{i\in F\setminus\{m\}} [(k+2)|A_i| - 2k] \\
&\quad + (k+2)|A_m| + m(|A_n| - |A_m|) - 2k \\
&\geq \sum_{i\in E\setminus\{m\}} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i\in E\setminus\{m\}} |A_i| + \sum_{i\in F\setminus\{m\}} [(k+2)|A_i| - 2k] \\
&\quad + (k+2)|A_m| - 2k \\
&\geq (k+2)|A| - 4(|E| + |F|)k^{t-2} \geq (k+2)|A| - 4k^{t-1}.
\end{aligned}$$

□

Proof of Theorem 1. If $j = k$, applying Corollary 6, we obtain $|2\cdot A + k\cdot A| \geq (k+2)|A| - 2k \geq (k+2)|A| - k^2 - k + 2$. We assume $j < k$. Without loss of generality we also assume that $\gcd(A) = 1$ and $0 \in A_1$. We have $|A_1| \geq \frac{|A|}{j} > 8k^{k-1}$. Let $m = \min\{1 \leq i \leq j \mid p \nmid u_i\}$. We distinguish two cases.

Case 1. $E = \emptyset$.

By Corrolary 6 and Lemma 7, we have

$$\begin{aligned}
|2\cdot A + k\cdot A| &= \sum_{i=1}^{j} |2\cdot A_i + k\cdot A| \\
&= \sum_{i=1}^{j} (|2\cdot A_i + k\cdot A_i| + |\Delta_{ii}|) \\
&= \sum_{i=1}^{j} |2\cdot X_i + k\cdot X_i| + \sum_{i=1}^{j} |\Delta_{ii}| \\
&\geq \sum_{i=1}^{j} [(k+2)|X_i| - 2k] + j(j-1) \\
&= (k+2)|A| - j(2k - j + 1) \\
&\geq (k+2)|A| - k^2 - k + 2.
\end{aligned}$$

Case 2. $E \neq \emptyset$.

We consider following subcases.

Case 2a. $m \in E$

By Lemma 11, we have $|\Delta_{mm}| \geq |A_1|$. Applying Lemma 12, we obtain

$$\begin{aligned}|2 \cdot A + k \cdot A| &\geq |2 \cdot A_m + k \cdot A| + |2 \cdot (A \setminus A_m) + k \cdot (A \setminus A_m)| \\ &= |2 \cdot A_m + k \cdot A_m| + |\Delta_{mm}| + (k+2)|A \setminus A_m| - 4k^{k-1} \\ &\geq (k+2)|A_m| - 4k^{k-1} + |A_1| + (k+2)|A \setminus A_m| - 4k^{k-1} \\ &> (k+2)|A|.\end{aligned}$$

Case 2b. $m \in F$.
If $|\Delta_{11}| \geq |A_1|$, we have

$$\begin{aligned}|2 \cdot A + k \cdot A| &\geq |2 \cdot A_1 + k \cdot A| + |2 \cdot (A \setminus A_1) + k \cdot (A \setminus A_1)| \\ &= |2 \cdot A_1 + k \cdot A_1| + |\Delta_{11}| + (k+2)|A \setminus A_m| - 4k^{k-1} \\ &\geq (k+2)|A_1| - 4k^{k-1} + |A_1| + (k+2)|A \setminus A_m| - 4k^{k-1} \\ &> (k+2)|A|.\end{aligned}$$

If $|\Delta_{11}| < |A_1|$, then by Lemma 10, we have $c_2(A_1) = 2$. Since $E \neq \emptyset$, there exists $s \in E$. By Lemma 11, we have $|\Delta_{ss}| \geq |A_m|$ if $p \mid u_s$ and $|\Delta_{ss}| \geq |A_1|$ if $p \nmid u_s$. Since $|A_1| \geq |A_m|$, we obtain $|\Delta_{ss}| \geq |A_m|$. We denote $A' = A \setminus (A_m \cup A_s)$. Applying Proposition 4, we obtain

$$\begin{aligned}|2 \cdot A + k \cdot A| &\geq |2 \cdot A_m + k \cdot A| + |2 \cdot A_s + k \cdot A| + |2 \cdot A' + k \cdot A'| \\ &\geq |2 \cdot A_m + k \cdot A_1| + |2 \cdot A_s + k \cdot A_s| + |\Delta_{ss}| + (k+2)|A'| - 4k^{k-1} \\ &\geq 2|A_m| + k|A_1| - 2k + (k+2)|A_s| - 4k^{k-1} + |A_m| + (k+2)|A'| - 4k^{k-1} \\ &= (k+2)|A| + |A_1| - 8k^{k-1} - 2k \\ &> (k+2)|A| - 2k\end{aligned}$$

This ends the proof.

4. The case $k = pq$

Lemma 13. *Let A be a finite set of integers such that $\gcd(A) = 1$. Let $k = pq$, where p and q are distinct odd prime numbers. Let $m = \min\{1 \leq i \leq j \mid p \nmid u_i\}$ and let $i \in E$.*

(i) *If $(u_2, k) = 1$, then*

$$|\Delta_{ii}| \geq \begin{cases} |A_1| & \text{if } i = 2 \\ |A_2| & \text{if } i \neq 2 \end{cases}$$

(ii) *If $(u_2, k) = p$, then*

$$|\Delta_{ii}| \geq \begin{cases} \min\{|A_2|, q|A_m|\} & \text{if } i = 1 \\ \min\{|A_1|, q|A_m|\} & \text{if } 1 < i < m \\ |A_2| & \text{if } i = m \\ \min\{|A_1|, |A_2|, q|A_m|\} & \text{if } i > m \end{cases}$$

Proof. (i) By Lemma 8, if $i \in E$, we have $2 \cdot \hat{X}_i = 2 \cdot \hat{X}_i + u_1 \neq 2 \cdot \hat{X}_i + u_2$. Otherwise, $k \mid |X_i|$, a contradiction. Thus, if $1 \in E$, we have

$$\begin{aligned} |\Delta_{11}| &= |(2 \cdot A_1 + k \cdot A) \setminus (2 \cdot A_1 + k \cdot A_1)| \geq |(2 \cdot X_1 + A_2) \setminus (2 \cdot X_1 + A_1)| \\ &\geq |A_2||(2 \cdot \hat{X}_1 + u_2) \setminus (2 \cdot \hat{X}_1 + u_1)| \geq |A_2|. \end{aligned}$$

Similarly, if $2 \in E$, we have $|\Delta_{11}| \geq |A_1|$.

Now, let $i \in E$ and $i \neq 1, 2$. Since $2 \cdot \hat{X}_i + u_1 \neq 2 \cdot \hat{X}_i + u_2$, we have that $2 \cdot \hat{X}_i + u_i \neq 2 \cdot \hat{X}_i + u_1$, in which case $|\Delta_{ii}| \geq |A_1|$ or $2 \cdot \hat{X}_i + u_i \neq 2 \cdot \hat{X}_i + u_2$, in which case $|\Delta_{ii}| \geq |A_2|$. In both cases $|\Delta_{ii}| \geq |A_2|$.

(ii) Let $1 \in E$. Then $2 \cdot \hat{X}_1 + u_1 \neq 2 \cdot \hat{X}_1 + u_2$ or $2 \cdot \hat{X}_1 + u_1 = 2 \cdot \hat{X}_1 + u_2$. If $2 \cdot \hat{X}_1 + u_1 \neq 2 \cdot \hat{X}_1 + u_2$, we obtain, as in (i), that $|\Delta_{11}| \geq |A_2|$. If $2 \cdot \hat{X}_1 + u_1 = 2 \cdot \hat{X}_1 + u_2$, by Lemma 8, we have that

$$2 \cdot \hat{X}_1 = \bigcup_{\beta \in I} (p \cdot \{0, 1, \ldots, q-1\} + \beta)$$

for some nonempty set $I \subset \mathbb{Z}/p\mathbb{Z}$. Moreover, $p \nmid u_m$, thus $I + u_m \neq I$ and $|(2 \cdot \hat{X}_1 + u_m) \setminus (2 \cdot \hat{X}_1 + u_1)| \geq q$. We obtain

$$\begin{aligned} |\Delta_{11}| &= |(2 \cdot A_1 + k \cdot A) \setminus (2 \cdot A_1 + k \cdot A_1)| \geq |(2 \cdot X_1 + A_m) \setminus (2 \cdot X_1 + A_1)| \\ &\geq |A_m||(2 \cdot \hat{X}_1 + u_m) \setminus (2 \cdot \hat{X}_1 + u_1)| \geq q|A_m|. \end{aligned}$$

Next, if $i < m$, we have that $p \mid u_i$ and $(k, u_i) = p$. As above, we have $2 \cdot \hat{X}_i + u_i \neq 2 \cdot \hat{X}_i + u_1$ or $2 \cdot \hat{X}_i + u_i = 2 \cdot \hat{X}_i + u_1$ and we obtain $|\Delta_{ii}| \geq |A_1|$ or $|\Delta_{ii}| \geq q|A_m|$. If $m \in E$, we have $p \nmid u_m$. Thus, $q \nmid u_m$, in which case $(k, u_m) = 1$, or $q \mid u_m$, in which case $(k, u_m - u_2) = 1$. Thus, $2 \cdot \hat{X_m} + u_m \neq 2 \cdot \hat{X_m} + u_1$ or $2 \cdot \hat{X_m} + u_m \neq 2 \cdot \hat{X_m} + u_2$. We have

$$|\Delta_{mm}| = |(2 \cdot X_m + A) \setminus (2 \cdot X_1 + A_m)| \geq |A_2|.$$

Finally, if $i > m$, we have $(k, u_i) = 1$ or $(k, u_i) = p$ or $(k, u_i) = q$. If $(k, u_i) = 1$, we have $|\Delta_{ii}| \geq |A_1|$. If $(k, u_i) = p$, we obtain $|\Delta_{ii}| \geq |A_1|$ or $|\Delta_{ii}| \geq q|A_m|$. If $(k, u_i) = q$, we have $(k, u_m - u_2) = 1$ and $|\Delta_{ii}| \geq |A_2|$.

□

Lemma 14. *Let A be a finite set of integers. If $k = pq$, where p and q are distinct odd primes, then*

$$|2 \cdot A + k \cdot A| \geq (k+2)|A| - 4k^{k-1}.$$

Proof. Let T be the set of integers t such that for every finite set $A \subset \mathbb{Z}$

$$|2 \cdot A + k \cdot A| \geq (t+2)|A| - 4k^{t-1}.$$

As in the proof of Lemma 12, we will use induction to prove $k \in T$. By Corollary 5, we have that $2 \in T$. Let us assume that $2 \leq t \leq k$ and $t - 1 \in T$. Let A be a finite set of integers. Without loss of generality we may assume that $\gcd(A) = 1$ and that $0 \in A_1$. We define $m = \min\{1 \leq i \leq j \mid p \nmid u_i\}$.

If $\sum_{i \in E} |\Delta_{ii}| \geq \sum_{i \in E} |A_i|$ the same proof holds as in Lemma 12. Let us assume that $\sum_{i \in E} |\Delta_{ii}| < \sum_{i \in E} |A_i|$. We define $n = \min\{i \in E \mid |\Delta_{ii}| < |A_i|\}$.

Case 1. $(u_2, k) = 1$. We have $2 \in F$. Otherwise, $2 \in E$ and by Lemma 13, we have $\sum_{i \in E} |\Delta_{ii}| \geq \sum_{i \in E} |A_i|$, a contradiction. Moreover, since $|\Delta_{ii}| \geq |A_2|$ for all

$i \in E$, we obtain that $1 \in E$ and $|\Delta_{11}| < |A_1|$. By Lemma 10, we have $c_2(A_1) = 2$. We obtain

$$\begin{aligned}
|2 \cdot A + k \cdot A| &= \sum_{i \in E} |2 \cdot A_i + k \cdot A| + \sum_{i \in F \setminus \{2\}} |2 \cdot A_i + k \cdot A| + |2 \cdot A_2 + k \cdot A| \\
&\geq \sum_{i \in E} (|2 \cdot A_i + k \cdot A_i| + |\Delta_{ii}|) + \sum_{i \in F \setminus \{2\}} |2 \cdot A_i + k \cdot A_i| \\
&\quad + |2 \cdot A_2 + k \cdot A_1| \\
&\geq \sum_{i \in E} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E} |\Delta_{ii}| + \sum_{i \in F \setminus \{2\}} [(k+2)|A_i| - 2k] \\
&\quad + 2|A_2| + k|A_1| - 2k \\
&\geq \sum_{i \in E} [(t+1)|A_i| - 4k^{t-2}] + \sum_{i \in E} |A_2| + \sum_{i \in F \setminus \{2\}} [(k+2)|A_i| - 2k] \\
&\quad + (k+1)|A_2| + |A_1| - 2k \\
&\geq (t+2)|A| - 4(|E| + |F|)k^{t-2} \geq (t+2)|A| - 4k^{t-1}.
\end{aligned}$$

Case 2. $(u_2, k) = p$. Thus $m \geq 3$. By Lemma 10, we have $c_2(A_n) = 2$. By Lemma 13, we have $|\Delta_{ii}| \geq |A_m|$ for all $i \in E$. In particular $m \neq n$. Similarly as in Lemma 12, we obtain $m > n$. We have

$$\begin{aligned}
|2 \cdot A + k \cdot A| = &\sum_{i \in E \setminus \{m\}} |2 \cdot A_i + k \cdot A| + \sum_{i \in F \setminus \{m\}} |2 \cdot A_i + k \cdot A| + |2 \cdot A_m + k \cdot A| \\
\geq &\sum_{i \in E \setminus \{m\}} (|2 \cdot A_i + k \cdot A_i| + |\Delta_{ii}|) + \sum_{i \in F \setminus \{m\}} |2 \cdot A_i + k \cdot A_i| \\
&+ |2 \cdot X_m + A| \\
\geq &\sum_{i \in E \setminus \{m\}} (t+1)|A_i| + \sum_{i \in E \setminus \{m\}} |\Delta_{ii}| + \sum_{i \in F \setminus \{m\}} (t+2)|A_i| + |2 \cdot X_m + A| \\
&- \Big(\sum_{i \in E \setminus \{m\}} 4k^{t-2} + \sum_{i \in F \setminus \{m\}} 2k\Big).
\end{aligned}$$

If $m \in F$, using Proposition 4, we obtain

$$\begin{aligned}
|2 \cdot X_m + A| \geq |2 \cdot X_m + A_n| &= |2 \cdot X_m + k \cdot X_n| \geq 2|X_m| + k|X_n| - 2k \\
&= (k+2)|A_m| + k(|A_n| - |A_m|) - 2k.
\end{aligned}$$

Thus

$$\begin{aligned}
|2 \cdot A + k \cdot A| &\geq \sum_{i \in E} (t+1)|A_i| + \sum_{i \in E} |\Delta_{ii}| + \sum_{i \in F \setminus \{m\}} (t+2)|A_i| \\
&\quad + (t+2)|A_m| + k(|A_n| - |A_m|) - (|E| 4k^{t-2} + |F| 2k) \\
&\geq (t+2)|A| - 4k^{t-1}.
\end{aligned}$$

Next, let us assume $m \in E$. We have

(5)
$$|2 \cdot A + k \cdot A| \geq \sum_{i \in E \setminus \{m\}} (t+1)|A_i| + \sum_{i \in E \setminus \{m\}} |\Delta_{ii}| + \sum_{i \in F} (t+2)|A_i| + |2 \cdot X_m + A| \\ - (\sum_{i \in E \setminus \{m\}} 4k^{t-2} + \sum_{i \in F} 2k).$$

If $|A_1| \leq q|A_m|$, using Lemma 13, we obtain that $1 \in E$ and

$$\sum_{i \in E} |\Delta_{ii}| \geq |A_2| + \sum_{i \in E, i \geq 2} |A_i|.$$

In particular, $|\Delta_{11}| < |A_1|$. Moreover, $2 \notin E$, otherwise, by Lemma 13, $|\Delta_{ii}| \geq |A_1|$ and $\sum_{i \in E} |\Delta_{ii}| \geq \sum_{i \in E} |A_i|$. Using the same argument as in the *Case 1*, we obtain

$$|2 \cdot A + k \cdot A| \geq (t+2)|A| - 4k^{t-1}.$$

We assume $|A_1| > q|A_m|$. Then $|A_n| > q|A_m|$. Otherwise, $n \geq 2$ and by Lemma 13, we have $|\Delta_{ii}| \geq |A_n|$, for all $i \in E$ and $\sum_{i \in E} |\Delta_{ii}| \geq \sum_{i \in E} |A_i|$, a contradiction. By Lemma 13, $|\Delta_{11}| \geq \min\{|A_2|, q|A_m|\}$, $|\Delta_{ii}| \geq q|A_m|$ for all $i \in E$ such that $1 < i < m$ and $|\Delta_{ii}| \geq |A_m|$ for all $i \in E$ such that $i \geq m$. We need to consider separately the cases $|\hat{X}_m| < p$ and $|\hat{X}_m| \geq p$. Moreover, the case $|\hat{X}_m| \geq p$, we will subdivided in three subcases: $p \leq |\hat{X}_m| < q$, $|\hat{X}_m| \geq p > q$ and $|\hat{X}_m| \geq q > p$. We will use that $m \leq q+1$.

Case 2a. $|\hat{X}_m| \geq p > q$. By Corollary 6, we have

$$\begin{aligned} |2 \cdot X_m + A| \geq |2 \cdot X_m + A_n| &\geq 2|X_m| + |\hat{X}_m||X_n| - 2k \\ &\geq (k+2)|A_m| + p(|A_n| - q|A_m|) - 2k \\ &\geq (t+2)|A_m| + (m-1)(|A_n| - q|A_m|) - 2k. \end{aligned}$$

If $n > 1$, by (5), we have

$$\begin{aligned} |2 \cdot A + k \cdot A| \geq & \sum_{i \in E \setminus \{m\}} (t+1)|A_i| + \sum_{i \in E \setminus \{m\}} |\Delta_{ii}| + \sum_{i \in F} (t+2)|A_i| \\ & + (t+2)|A_m| + (m-1)(|A_n| - q|A_m|) \\ & - ((|E|-1)4k^{t-2} + (|F|+1)2k) \\ \geq & (t+2)|A| - 4k^{t-1}. \end{aligned}$$

If $n = 1$, then $c_2(A_1) = 2$. We need to consider following subcases.

If $2 \in E$, by Lemma 13, we have that $|\Delta_{11}| \geq \min\{|A_2|, q|A_m|\}$ and $|\Delta_{22}| \geq \min\{|A_1|, q|A_m|\}$, so the above proof holds.

If $2 \in F$, using Proposition 4, we obtain

$$|2 \cdot X_2 + A| \geq |2 \cdot X_2 + A_1| = |2 \cdot X_2 + k \cdot X_1| \geq 2|A_2| + k|A_1| - 2k$$

so

$$
\begin{aligned}
|2\cdot A+k\cdot A| &= \sum_{i\in E\setminus\{m\}}|2\cdot A_i+k\cdot A|+\sum_{i\in F\setminus\{2\}}|2\cdot A_i+k\cdot A|\\
&\quad+|2\cdot A_2+k\cdot A|+|2\cdot A_m+k\cdot A|\\
&\geq \sum_{i\in E\setminus\{m\}}(|2\cdot A_i+k\cdot A_i|+|\Delta_{ii}|)+\sum_{i\in F\setminus\{2\}}|2\cdot A_i+k\cdot A_i|\\
&\quad+|2\cdot X_2+A|+|2\cdot X_m+A|\\
&\geq \sum_{i\in E\setminus\{m\}}(t+1)|A_i|+\sum_{i\in E\setminus\{m\}}|\Delta_{ii}|+\sum_{i\in F\setminus\{2\}}(t+2)|A_i|\\
&\quad+2|A_2|+k|A_1|+(t+2)|A_m|+(m-1)(|A_1|-q|A_m|)\\
&\quad-((|E|-1)4k^{t-2}+(|F|+1)2k)\\
&\geq (t+2)|A|-4k^{t-1}.
\end{aligned}
$$

Case 2b. $|\hat{X}_m|\geq q>p$. Similarly as in previous case, we obtain

$$
\begin{aligned}
|2\cdot X_m+A| \geq |2\cdot X_m+A_n| &\geq 2|X_m|+|\hat{X}_m||X_n|-2k\\
&\geq (k+2)|A_m|+q(|A_n|-p|A_m|)-2k\\
&\geq (t+2)|A_m|+(m-1)(|A_n|-q|A_m|)-2k
\end{aligned}
$$

and

$$|2\cdot A+k\cdot A|\geq (t+2)|A|-4k^{t-1}.$$

Case 2c. $|\hat{X}_m|<p$. We have $|\hat{X}_m|+m-1<p+q\leq pq=k$. Let $L=\{1\leq i\leq m-1 \mid (u_i-u_m,k)\neq 1\}$. If $L=\emptyset$, then $(u_1-u_m,k)=\cdots=(u_{m-1}-u_m,k)=1$. Using Lemma 3, we obtain

$$|2\cdot \hat{X}_m+\{0,u_1-u_m,\ldots,u_s-u_m\}|\geq |\hat{X}_m|+s-1,\ \text{for all } 1\leq s\leq m-1.$$

Let $s=1$. We have

$$|(2\cdot \hat{X}_m+u_1)\setminus(2\cdot \hat{X}_m+u_m)|\geq 1$$

and

$$|(2\cdot X_m+A_1)\setminus(2\cdot X_m+A_m)|\geq |A_1|.$$

Now, let $2\leq s\leq m-1$ such that

$$|(2\cdot X_m+(A_1\cup A_2\cup\ldots\cup A_{s-1}))\setminus(2\cdot X_m+A_m)|\geq |A_1|+|A_2|+\cdots+|A_{s-1}|.$$

We have

$$|(2\cdot \hat{X}_m+\{u_1,\ldots,u_{s-1}\})\setminus(2\cdot \hat{X}_m+u_m)|\geq s-1$$

and

$$|(2\cdot \hat{X}_m+\{u_1,\ldots,u_{s-1},u_t\})\setminus(2\cdot \hat{X}_m+u_m)|\geq s,$$

so

$$|(2\cdot X_m+(A_1\cup A_2\cup\ldots\cup A_s)\setminus(2\cdot X_m+A_m)|\geq |A_1|+|A_2|+\cdots+|A_s|.$$

We have

$$|(2\cdot X_m+(A_1\cup A_2\cup\ldots\cup A_{m-1})\setminus(2\cdot X_m+A_m)|\geq |A_1|+|A_2|+\cdots+|A_{m-1}|.$$

Hence,

$$\begin{aligned}|\Delta_{mm}| &= |(2\cdot A_m + k\cdot A)\setminus(2\cdot A_m + k\cdot A_m)|\\ &= |(2\cdot X_m + A)\setminus(2\cdot X_m + X_m)|\\ &\geq |(2\cdot X_m + (A_1\cup A_2\cup\ldots\cup A_m))\setminus(2\cdot X_m + A_m)|\\ &\geq |A_1| + |A_2| + \cdots + |A_{m-1}|\end{aligned}$$

and

$$\begin{aligned}|2\cdot A_m + k\cdot A| &\geq |2\cdot A_m + k\cdot A_m| + |\Delta_{mm}|\\ &\qquad\geq (t+1)|A_m| + |A_1| + |A_2| + \cdots + |A_{m-1}| - 4k^{t-2}.\end{aligned}$$

Using (5), we obtain

$$\begin{aligned}|2\cdot A + k\cdot A| \geq{}& \sum_{i\in E}(t+1)|A_i| + \sum_{i\in E\setminus\{m\}}|\Delta_{ii}| + \sum_{i\in F}(t+2)|A_i|\\ &+ |A_1| + |A_2| + \cdots + |A_{m-1}| - (|E|4k^{t-2} + |F|2k)\\ \geq{}& (t+2)|A| - 4k^{t-1}.\end{aligned}$$

Now, let us assume that $L \neq \emptyset$. Thus there exists $1 \leq l \leq m-1$ such that $(u_l - u_m, k) \neq 1$. Since $p \mid u_l$ and $p \nmid u_m$, we obtain that $(u_l - u_m, k) = q$. Thus $|L| = 1$. Since $|X_m| < p$, by Lemma 8, we have $|\hat{X}_m + (u_l - u_m)\setminus \hat{X}_m| \geq 1$. Then, using Lemma 3 and Lemma 9, we obtain

$$|2\cdot\hat{X}_m + \{0, u_1 - u_m, \ldots, u_s - u_m\}| \geq |\hat{X}_m| + s - 1, \text{ for all } 1 \leq s \leq m-1.$$

Similarly as in the case $L = \emptyset$, we have

$$|(2\cdot X_m + (A_1\cup A_2\cup\ldots\cup A_m)\setminus(2X_m + A_m)| \geq |A_1| + |A_2| + \cdots + |A_{m-1}|$$

and

$$|2\cdot A + k\cdot A| \geq (t+2)|A| - 4k^{t-1}.$$

Case 2d. $p \leq |\hat{X}_m| < q$. Let $(X_m)_q = \{x (\mathrm{mod}\ q) \mid q \in X_m\}$. Then $|(X_m)_q| \leq |\hat{X}_m| < q$. Moreover, $(u_i, q) = 1$, for $2 \leq i \leq m-1$ and $|\{u_i (\mathrm{mod}\ q) \mid 2 \leq i \leq m-1\}| = m-2$, so by Lemma 3

$$|2\cdot(X_m)_q + \{u_1(\mathrm{mod}\ q), u_2(\mathrm{mod}\ q), \ldots, u_t(\mathrm{mod}\ q)\}| \geq \min\{q, |(X_m)_q| + s - 1\}$$

for all $1 \leq s \leq m-1$. Similarly, as in the previous case, we have

$$|(X_m + (A_1\cup A_2\cup\cdots\cup A_m)\setminus(X_m + A_1)| \geq |A_2| + \cdots + |A_r|,$$

where $r = \min\{m-1, q+1-(X_m)_q\}$. We obtain

$$\begin{aligned}|2\cdot X_m + A| &\geq |2\cdot X_m + A_1| + |(X_m + (A_1\cup A_2\cup\cdots\cup A_m)\setminus(X_m + A_1)|\\ &\geq c_2(A_1)|X_m| + |\hat{X}_m||A_1| + |A_2| + \cdots + |A_r| - 2k.\end{aligned}$$

We have two subcases: $|\Delta_{11}| \geq |A_1|$ or $|\Delta_{11}| < |A_1|$ and $c_2(A_1) = 2$. In both subcases, using Lemma 13, we obtain

$$
\begin{aligned}
|2\cdot A+k\cdot A| &= \sum_{i\in E\setminus\{m\}}|2\cdot A_i+k\cdot A|+|2\cdot A_m+k\cdot A|+\sum_{i\in F}|2\cdot A_i+k\cdot A|\\
&\geq \sum_{i\in E\setminus\{m\}}|2\cdot A_i+k\cdot A_i|+\sum_{i\in E\setminus\{m\}}|\Delta_{ii}|+|2\cdot X_m+A|\\
&\quad+\sum_{i\in F,i\leq m-1}|2\cdot A_i+k\cdot A_n|+\sum_{i\in F,i>m}|2\cdot A_i+k\cdot A_i|\\
&\geq \sum_{i\in E\setminus\{m\}}((t+1)|A_i|-4k^{t-1})\\
&\quad+|\Delta_{11}|+\sum_{i\in E,2\leq i\leq m-1}q|A_m|+\sum_{i\in E,m+1\leq i\leq j}|A_m|\\
&\quad+\sum_{i\in F,i\leq m-1}(2|A_i|+k|A_n|-2k)+\sum_{i\in F,i>m}((k+2)|A_i|-2k)\\
&\quad+c_2(A_1)|X_m|+|\hat{X}_m||A_1|+|A_2|+\cdots+|A_r|-2k\\
&\geq \sum_{i\in E\setminus\{m\}}(t+1)|A_i|+\sum_{i\in E,2\leq i\leq m-1}q|A_m|+\sum_{i\in E,m+1\leq i\leq j}|A_m|\\
&\quad+\sum_{i\in F,i\leq m-1}(2|A_i|+k|A_n|)+\sum_{i\in F,i>m}(k+2)|A_i|\\
&\quad+2|X_m|+|\hat{X}_m||A_1|+|A_2|+\cdots+|A_r|-2k\\
&\quad-((|E|-1)4k^{t-2}+(|F|+1)2k)\\
&\geq \sum_{i\in E\setminus\{m\}}(t+2)|A_i|+\sum_{i\in F}(t+2)|A_i|+2|A_m|+(|\hat{X}_m|-m+r)|A_1|\\
&\quad+(m-2)q|A_m|-((|E|-1)4k^{t-2}+(|F|+1)2k)\\
&\geq \sum_{i\in(E\cup F)\setminus\{m\}}(t+2)|A_i|+2|A_m|+(|\hat{X}_m|+r-2)q|A_m|\\
&\quad-((|E|-1)4k^{t-2}+(|F|+1)2k).
\end{aligned}
$$

By the definition of r, we have

$$
\begin{aligned}
|\hat{X}_m|+r-2 &\geq \min\{|\hat{X}_m|+m-3,|\hat{X}_m|+q-1-(X_m)_q\}\\
&\geq \min\{|\hat{X}_m|+m-3,q-1\}\geq p.
\end{aligned}
$$

Thus

$$|2\cdot A+k\cdot A|\geq(t+2)|A|-4k^{t-1}.$$

$\square$

Proof of Theorem 2. If $j=k$, applying Corollary 6, we obtain $|2\cdot A+k\cdot A|\geq(k+2)|A|-2k\geq(k+2)|A|-k^2-k+2$. We assume $j<k$. Without loss of generality we also assume that $\gcd(A)=1$ and $0\in A_1$. We have $|A_1|\geq\frac{|A|}{j}>8k^{k-1}$. Let $m=\min\{1\leq i\leq j\mid p\nmid u_i\}$.

The proof in the case $E=\emptyset$ is the same as the proof of this case in Theorem 1. We assume $E\neq\emptyset$.

If $|\Delta_{11}| \geq |A_1|$, we have

$$\begin{aligned}|2\cdot A+k\cdot A| &\geq |2\cdot A_1+k\cdot A|+|2\cdot(A\setminus A_1)+k\cdot(A\setminus A_1)|\\ &= |2\cdot A_1+k\cdot A_1|+|\Delta_{11}|+(k+2)|A\setminus A_m|-4k^{k-1}\\ &\geq (k+2)|A_1|-4k^{k-1}+|A_1|+(k+2)|A\setminus A_m|-4k^{k-1}\\ &> (k+2)|A|.\end{aligned}$$

We assume $|\Delta_{11}| < |A_1|$. Then by Lemma 10, we have $c_2(A_1) = 2$. We consider following cases.

Case 1. $(u_2, k) = 1$

Let $2 \in F$. Since $E \neq \emptyset$, there exists $s \in E$. By Lemma 13, we have $|\Delta_{ss}| \geq |A_2|$. We denote $A' = A \setminus (A_2 \cup A_s)$. We obtain

$$\begin{aligned}|2\cdot A+k\cdot A| &\geq |2\cdot A_2+k\cdot A|+|2\cdot A_s+k\cdot A|+|2\cdot A'+k\cdot A'|\\ &\geq |2\cdot A_2+k\cdot A_1|+|2\cdot A_s+k\cdot A_s|+|\Delta_{ss}|+(k+2)|A'|-4k^{k-1}\\ &\geq 2|A_2|+k|A_1|-2k+(k+2)|A_s|-4k^{k-1}+|A_2|+(k+2)|A'|-4k^{k-1}\\ &\geq (k+2)|A|+|A_1|-8k^{k-1}-2k\\ &> (k+2)|A|-2k\end{aligned}$$

If $2 \in E$, then by Lemma 13, we have $|\Delta_{22}| \geq |A_1|$. Thus

$$\begin{aligned}|2\cdot A+k\cdot A| &\geq |2\cdot A_2+k\cdot A|+|2\cdot(A\setminus A_2)+k\cdot(A\setminus A_2)|\\ &= |2\cdot A_2+k\cdot A_2|+|\Delta_{22}|+(k+2)|A\setminus A_2|-4k^{k-1}\\ &\geq (k+2)|A_2|-4k^{k-1}+|A_1|+(k+2)|A\setminus A_2|-4k^{k-1}\\ &\geq (k+2)|A|+|A_1|-8k^{k-1}\\ &> (k+2)|A|.\end{aligned}$$

Case 2. $(u_2, k) = p$. We consider the following subcases.

Case 2a. $m \in F$. Since $E \neq \emptyset$, there exists $s \in E$. By Lemma 13, we have $|\Delta_{ss}| \geq |A_m|$. We denote $A' = A \setminus (A_m \cup A_s)$. We obtain

$$\begin{aligned}|2\cdot A+k\cdot A| &\geq |2\cdot A_m+k\cdot A|+|2\cdot A_s+k\cdot A|+|2\cdot A'+k\cdot A'|\\ &\geq |2\cdot A_m+k\cdot A_1|+|2\cdot A_s+k\cdot A_s|+|\Delta_{ss}|+(k+2)|A'|-4k^{k-1}\\ &\geq 2|A_m|+k|A_1|-2k+(k+2)|A_s|-4k^{k-1}+|A_m|+(k+2)|A'|-4k^{k-1}\\ &\geq (k+2)|A|+|A_1|-8k^{k-1}-2k\\ &> (k+2)|A|-2k\end{aligned}$$

Case 2b. $m \in E$. Here we will consider separate cases when $|\hat{X}_m| \geq p$ and $|\hat{X}_m| < p$. Moreover, the case $|\hat{X}_m| \geq p$ we will divide in two subcases: $|A_1| \leq q|A_m|$ and $|A_1| > q|A_m|$.

First we assume that $|\hat{X}_m| \geq p$ and $|A_1| \leq q|A_m|$.

Let $2 \in F$. By Lemma 13, we have $|\Delta_{mm}| \geq |A_2|$. We denote $A' = A \setminus (A_2 \cup A_m)$. We obtain

$$\begin{aligned}
|2\cdot A + k\cdot A| &\geq |2\cdot A_2 + k\cdot A| + |2\cdot A_m + k\cdot A| + |2\cdot A' + k\cdot A'| \\
&\geq |2\cdot A_2 + k\cdot A_1| + |2\cdot A_m + k\cdot A_m| + |\Delta_{mm}| + (k+2)|A'| - 4k^{k-1} \\
&\geq 2|A_2| + k|A_1| - 2k + (k+2)|A_m| - 4k^{k-1} + |A_2| + (k+2)|A'| - 4k^{k-1} \\
&\geq (k+2)|A| + |A_1| - 8k^{k-1} - 2k \\
&> (k+2)|A| - 2k
\end{aligned}$$

If $2 \in E$, by Lemma 13, we have $|\Delta_{22}| \geq |A_1|$. Thus

$$\begin{aligned}
|2\cdot A + k\cdot A| &\geq |2\cdot A_2 + k\cdot A| + |2\cdot (A\setminus A_2) + k\cdot (A\setminus A_2)| \\
&= |2\cdot A_2 + k\cdot A_2| + |\Delta_{22}| + (k+2)|A\setminus A_2| - 4k^{k-1} \\
&\geq (k+2)|A_2| - 4k^{k-1} + |A_1| + (k+2)|A\setminus A_2| - 4k^{k-1} \\
&\geq (k+2)|A| + |A_1| - 8k^{k-1} \\
&> (k+2)|A|.
\end{aligned}$$

Next we assume that $|\hat{X}_m| \geq p$ and $|A_1| > q|A_m|$. By Corollary 6, we have

$$|2\cdot X_m + A| \geq |2\cdot X_m + A_1| \geq 2|X_m| + |\hat{X}_m||X_1| - 2k.$$

If $|\hat{X}_m| > p$, we obtain

$$|2\cdot X_m + A| \geq |2\cdot X_m + A_1| \geq 2|A_m| + (p+1)|A_1| - 2k. \tag{6}$$

If $|\hat{X}_m| = p$, by Lemma 8, we have $|(2\cdot \hat{X}_m + u_2)\setminus(2\cdot \hat{X}_m + u_1)| \geq 1$ and

$$|(2\cdot X_m + A_2)\setminus(2\cdot X_m + A_1)| \geq |A_2||(2\cdot \hat{X}_m + u_2)\setminus(2\cdot \hat{X}_m + u_1)| \geq |A_2|.$$

Thus

$$\begin{aligned}
|2\cdot X_m + A| &\geq |2\cdot X_m + A_1| + |(2\cdot X_m + A_2)\setminus(2\cdot X_m + A_1)| \\
&\geq |2\cdot X_m + A_1| + |A_2| \geq 2|A_m| + p|A_1| + |A_2| - 2k.
\end{aligned} \tag{7}$$

Now, let $2 \in F$. We denote $A' = A \setminus (A_2 \cup A_m)$. We have

$$\begin{aligned}
|2\cdot A + k\cdot A| &\geq |2\cdot A_2 + k\cdot A| + |2\cdot A_m + k\cdot A| + |2\cdot A' + k\cdot A'| \\
&\geq |2\cdot A_2 + k\cdot A_1| + |2\cdot X_m + A| + (k+2)|A'| - 4k^{k-1} \\
&\geq 2|A_2| + k|A_1| - 2k + (k+2)|A_m| + |A_2| - 2k + (k+2)|A'| - 4k^{k-1} \\
&\geq (k+2)|A| + |A_1| - 8k^{k-1} - 2k \\
&> (k+2)|A| - 2k
\end{aligned}$$

If $2 \in E$, by Lemma 13, we have $|\Delta_{22}| \geq q|A_m|$. Thus, if $A' = A \setminus (A_2 \cup A_m)$, then

$$\begin{aligned}
|2 \cdot A + k \cdot A| &\geq |2 \cdot A_2 + k \cdot A| + |2 \cdot A_m + k \cdot A| + |2 \cdot A' + k \cdot A'| \\
&\geq |2 \cdot A_2 + k \cdot A_2| + |\Delta_{22}| + |2 \cdot X_m + A| + (k+2)|A'| - 4k^{k-1} \\
&\geq (k+2)|A_2| - 4k^{k-1} + q|A_m| + 2|A_m| + p|A_1| + |A_2| - 2k \\
&\quad + (k+2)|A'| - 4k^{k-1} \\
&\geq (k+2)|A_2| + (k+2)|A'| + q|A_m| + 2|A_m| + (p-1)q|A_m| + |A_1| \\
&\quad - 8k^{k-1} - 2k \\
&\geq (k+2)|A| + |A_1| - 8k^{k-1} - 2k \\
&> (k+2)|A| - 2k
\end{aligned}$$

Finally, we assume that $|\hat{X}_m| < p$. By Lemma 8, we have $|(2 \cdot \hat{X}_m + u_1) \setminus (2 \cdot \hat{X}_m + u_m)| \geq 1$ and

$$|\Delta_{mm}| \geq |(2 \cdot X_m + A_1) \setminus (2 \cdot X_m + A_m)| \geq |A_1||(2 \cdot \hat{X}_m + u_2) \setminus (2 \cdot \hat{X}_m + u_1)| \geq |A_1|.$$

Thus

$$\begin{aligned}
|2 \cdot A + k \cdot A| &\geq |2 \cdot A_m + k \cdot A| + |2 \cdot (A \setminus A_m) + k \cdot (A \setminus A_m)| \\
&= |2 \cdot A_m + k \cdot A_m| + |\Delta_{mm}| + (k+2)|A \setminus A_m| - 4k^{k-1} \\
&\geq (k+2)|A_m| - 4k^{k-1} + |A_1| + (k+2)|A \setminus A_m| - 4k^{k-1} \\
&\geq (k+2)|A| + |A_1| - 8k^{k-1} \\
&> (k+2)|A|.
\end{aligned}$$

This ends the proof.

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